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ON SIEGEL-EISENSTEIN SERIES OF DEGREE 2 FOR LOW WEIGHTS

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1. INTRODUCTION

First we recall the case of elliptic modular forms. Let

$$\Gamma(N)=\{\gamma\in SL(2,\mathbb{Z})\mid\gamma\equiv 1_2 \bmod N\}$$

be the principal congruence subgroup of $SL(2,\mathbb{Z})$ of level N, and $M_k(\Gamma(N))$, $S_k(\Gamma(N)$ the space of modular forms and cusp forms respectively, of weight k with respect to $\Gamma(N)$. Then if $k\geq 2$ we can calculate $\dim S_k(\Gamma(N))$ by using the Riemann-Roch theorem. However $\dim S_1(\Gamma(N))$ is not yet known for general N.

On the othere hand, the complement space of $S_k(\Gamma(N))$ in $M_k(\Gamma(N))$ is easier to handle even in low weight cases. Asuume $N\geq 3$, we set

$$\mathcal{E}_k(\Gamma(N))=M_k(\Gamma(N))/S_k(\Gamma(N)).$$

It is well-known that $\mathcal{E}_k$ is generated by Eisenstein series. Put

$$E_{\Gamma(N)}^k(z)=\sum_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma(N)_\infty\backslash\Gamma(N)}(cz+d)^{-k} \tag{1.1}$$

with $\Gamma(N)_\infty=\{\left(\begin{smallmatrix}a&b\\0&d\end{smallmatrix}\right)\in\Gamma(N)\}$, which converges if $k\geq 3$ and $E_{\Gamma(N)}^k(z)\in M_k(\Gamma(N))$.

If $k\geq 3$ we have

$$M_k(\Gamma(N))=S_k(\Gamma(N))\oplus\left\langle E_{\Gamma(N)}^k|_k\gamma\;\middle|\;\gamma\in SL(2,\mathbb{Z})\right\rangle_{\mathbb{C}},$$

and $\dim\mathcal{E}_k(\Gamma(N))$ equals to the number of cusps of $\Gamma(N)\backslash\mathfrak{H}$ i.e.

$$\dim\mathcal{E}_k(\Gamma(N))=\frac{1}{2}N^2\prod_{p|N}(1-p^{-2}),\quad(k\geq 3).$$

More precisely let $\{\gamma_1,\dots,\gamma_r\}$ be a representative set of $\Gamma(N)\backslash SL(2,\mathbb{Z})/SL(2,\mathbb{Z})_\infty$, then $\{E_{\Gamma(N)}^k|_k\gamma_i^{-1}\}_i$ form a basis of $\mathcal{E}_k(\Gamma(N))$.

In the case of low weights i.e. $k=1,2$, the right-hand side of (1.1) does not converge. To avoid this problem, Hecke ([He]) considered the following modified Eisenstein series:

$$E_{\Gamma(N)}^k(z,s)=\sum_{\Gamma(N)_\infty\backslash\Gamma(N)}(cz+d)^{-k}|cz+d|^{-2s}, \tag{1.2}$$

with $z \in \mathfrak{H}$ and $s \in \mathbb{C}$. Then the right-hand side converges for $2\,\mathrm{Re}(s) + k > 2$. The important fact is that, for fixed z, this series has a meromorphic continuation to whole s-plane. Put $E^k_{\Gamma(N)}(z) = E^k_{\Gamma(N)}(z, 0)$ then $E_k|_k\gamma(z) = E_k(z)$ for all $\gamma \in \Gamma(N)$.

Consider the case of weight 2. Then $E^2_{\Gamma(N)}(z)$ is *not* holomorphic in z. However

$$E^2_{\Gamma(N)}|_k\gamma - E^2_{\Gamma(N)} \in M_2(\Gamma(N)), \quad \forall\gamma \in SL(2,\mathbb{Z}),$$

and $\{E^2_{\Gamma(N)}|_k\gamma_i^{-1} - E^2_{\Gamma(N)}|_k\gamma_1^{-1}\}_{i\geq 2}$ form a basis of $\mathcal{E}_2(\Gamma(N))$, i.e.

$$\dim \mathcal{E}_2(\Gamma(N)) = \{\text{number of the cusps}\} - 1.$$

If $k = 1$, we have $E^1_{\Gamma(N)}(z) \in M_1(\Gamma(N))$. In this case $\{E^1_{\Gamma(N)}|_k\gamma_i\}_i$ has many linear relations and we have

$$\dim \mathcal{E}_1(\Gamma(N)) = \frac{1}{2}\{\text{number of cusps}\}.$$

In this report, we study the analogue theory of Eisenstein series for Siegel modular forms.

2. Notation and setting

Notation

- $\mathfrak{H}_g = \{Z \in M_g(\mathbb{C}) \mid {}^tZ = Z,\ \mathrm{Im}(Z) > 0\}$.
- $\Gamma^g = Sp(g,\mathbb{Z}) = \{\gamma \in GL(2g,\mathbb{Z}) \mid {}^t\gamma J_g\gamma = J_g\}$, $J_g = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}$.
- For $\gamma \in \Gamma^g$, g by g matrices $A_\gamma, \ldots, D_\gamma$ are defined by $\gamma = \begin{pmatrix} A_\gamma & B_\gamma \\ C_\gamma & D_\gamma \end{pmatrix}$.
- $\Gamma^g_0(N) = \{\gamma \in \Gamma^g \mid C_\gamma \equiv 0 \bmod N\}$.
- $\Gamma^g(N) = \{\gamma \in \Gamma^g \mid \gamma \equiv 1_{2g} \bmod N\}$.

We define the space of Siegel modular forms of weight k with respect to $\Gamma^g(N)$ by

$$M_k(\Gamma^g(N)) = \{f\colon \mathfrak{H}_g \xrightarrow{\text{hol}} \mathbb{C} \mid f|_k\gamma = f,\ \forall\gamma \in \Gamma^g(N)\}$$

with $f|_k\gamma(Z) = \det(C_\gamma Z + D_\gamma)^{-k} f(\gamma\langle Z\rangle)$, $\gamma\langle Z\rangle = (A_\gamma Z + B_\gamma)(C_\gamma Z + D_\gamma)^{-1}$. If $g = 1$ we also require the holomorphic condition at each cusp.

For a Dirichlet character ψ modulo N, we set

$$M_k(\Gamma^g_0(N), \psi) = \{f \in M_k(\Gamma^g(N)) \mid f|_k\gamma = \psi(\det D_\gamma)f,\ \forall\gamma \in \Gamma^g_0(N)\}.$$

Now we define the Siegel-Eisenstein series. For $\Gamma \subset Sp(g,\mathbb{Z})$, put $\Gamma_\infty = \{\gamma \in \Gamma \mid C_\gamma = 0\}$. Let ψ be a Dirichlet character with $\psi(-1) = (-1)^k$. Then we define

$$E^k_{N,\psi}(Z,s) = \sum_{\gamma \in \Gamma^g_0(N)_\infty \backslash \Gamma^g_0(N)} \psi(\det D_\gamma)\det(C_\gamma Z + D_\gamma)^{-k}|\det(C_\gamma Z + D_\gamma)|^{-2s}. \tag{2.1}$$

The right-hand side converges absolutely and uniformly on $\mathfrak{H}_g$ for $2\,\mathrm{Re}(s)+k>g+1$. In particular if $k\geq g+2$, $E^k_{N,\psi}(Z):=E^k_{N,\psi}(Z,0)\in M_k(\Gamma^g_0(N),\bar{\psi})$.

Remark. In the case of elliptic modular forms in Introduction, we consider the Eisenstein series with respect to the principal congruence subgroup $\Gamma(N)$. However since

$$\begin{aligned}E^k_{\Gamma^g(N)}(Z,s)&=\sum_{\gamma\in\Gamma^g(N)_\infty\backslash\Gamma^g(N)}\det(C_\gamma Z+D_\gamma)^{-k}|\det(C_\gamma Z+D_\gamma)|^{-2s}\\&=\frac{2}{\phi(N)}\sum_{\psi(-1)=(-1)^k}E^k_{N,\psi}(Z,s),\quad(\phi\text{ is Euler's function,})\end{aligned}$$

it suffices to consider the Eisenstein series with respect to $\Gamma^g_0(N)$ with Dirichlet characters.

Let $C_0(f)$ be the constant term of the Fourier expansion of $f\in M_k(\Gamma^g(N))$, $L_k(\Gamma^g(N))=\{f\in M_k(\Gamma^g(N))\mid C_0(f|_k\gamma)=0,\ \forall\gamma\in\Gamma^g\}$. We put

$$\begin{aligned}\mathcal{E}_k(\Gamma^g(N))&=M_k(\Gamma^g(N))/L_k(\Gamma^g(N)),\\\mathcal{E}_k(\Gamma^g_0(N),\psi)&=M_k(\Gamma^g_0(N),\psi)/L_k(\Gamma^g(N))\cap M_k(\Gamma^g_0(N),\psi).\end{aligned}$$

Then it is easy to see that

Proposition 2.1. *Let $\{\gamma_\lambda\}_\lambda$ be a representative set of $\Gamma(N)\backslash\Gamma^g/\Gamma^g_\infty$. Then $\{E^k_{\Gamma^g(N)}|_k\gamma_\lambda^{-1}\}_\lambda$ form a basis of $\mathcal{E}_k(\Gamma^g(N))$. In particular for $g=2$, $N=p$ an odd prime and $k\geq 4$, we have*

$$\dim\mathcal{E}_k(\Gamma^2(p))=\frac{1}{2}(p^4-1).$$

3. Problems

In the rest of this report, we consider the low weight case. First we recall the following famous fact by Langlands [La]:

Theorem 3.1. *$E^k_{N,\psi}(Z,s)$ has a meromorphic continuation to whole s-plane.*

Now there are following natural three questions:

Q1 For each $Z\in\mathfrak{H}_g$, $E^k_{N,\psi}(Z,s)$ is regular at $s=0$?
Q2 $E^k_{N,\psi}(Z,0)$ is holomorphic in Z?
Q3 Calculate the dimension of $\mathcal{E}_k(\Gamma^g(N))$ (or $\mathcal{E}_k(\Gamma^g_0(N),\psi)$).

These questions are first raised and solved by G. Shimura in [Sh2] except for Q3. Instead of that, he considered the algebraicity of the Fourier coefficients of $E^k_{N,\psi}(Z)$, which is an important number theoretical question. However the result of [Sh2] is not sufficient to answer our Q3, because Shimura considered there only the Fourier expansion of $E^k_{N,\psi}|_kJ_g(Z,s)$, thus we can get no information for other cusps. Hence we have to study the behavior of $E^k_{N,\psi}$ at other cusps, in particular the Fourier expansion of $E^k_{N,\psi}(Z,s)$.

4. Fourier expansions of Eisenstein series

Let us forcus our problem to the case of $g = 2$ and $N = p$ an odd prime number. Let $\mathrm{Sym}^g(\mathbb{Z})^*$ be the dual lattice of $\mathrm{Sym}^g(\mathbb{Z})$ with respect to trace form. Then

$$\mathrm{Sym}^g(\mathbb{Z}) = \{h = (h_{ij}) \mid h_{ii} \in \mathbb{Z},\ 2h_{ij} \in \mathbb{Z}\ (i \neq j)\}.$$

Put $\mathbf{e}(X) = e^{2\pi i \operatorname{Tr}(X)}$ for a square matrix X, $A[B] := {}^tBAB$. We set $\Lambda_2 = \{{}^t(q_1, q_2) \in \mathbb{Z}^2 \mid (q_1, q_2) = 1\}$. Then the Fourier expansion of $E^k_{p,\psi}$ is give by (cf. [Ma, pp. 301-302])

$$\begin{aligned} E^k_{p,\psi}(Z, s) = 1 + \sum_{m\in\mathbb{Z}} \sum_{\left(\begin{smallmatrix} q_1 \\ q_2 \end{smallmatrix}\right)\in\Lambda_2/\{\pm1\}} & S_1(\psi, m, k+2s)\xi_1(Y[\left(\begin{smallmatrix} q_1 \\ q_2 \end{smallmatrix}\right)], m; k+s, s)\,\mathbf{e}(m\begin{pmatrix} q_1^2 & q_1q_2 \\ q_1q_2 & q_2^2 \end{pmatrix}X) \\ + \sum_{h\in\mathrm{Sym}^2(\mathbb{Z})^*} & S_2(\psi, h, k+2s)\xi_2(Y, h; k+s, s)\,\mathbf{e}(hX). \end{aligned} \tag{4.1}$$

We shall explain the notation. The function ξ_g is called the hypergeometric function defined by

$$\xi_g(Y, h; \alpha, \beta) = \int_{\mathrm{Sym}^g(\mathbb{R})} \det(X + iY)^{-\alpha} \det(X - iY)^{-\beta}\mathbf{e}(-hX)\, dX,$$

with $h \in \mathrm{Sym}^g(\mathbb{R})$, $\mathrm{Sym}^g(\mathbb{R}) \ni Y > 0$ and $\alpha, \beta \in \mathbb{C}$. This function is studied deeply by Shimura in [Sh1]. Roughly speaking, $\xi_g(Y, h, \alpha, \beta)$ is decomposed into the Γ-factor part and the entire function part on α and β. The explicit formula is as follows. For $\operatorname{sgn} h = (p, q, r)$,

$$\begin{aligned} \xi_g(Y, h; \alpha, \beta) &= i^{g(\beta-\alpha)}2^*\pi^*\Gamma_r(\alpha + \beta - \frac{g+1}{2})\Gamma_{g-q}(\alpha)^{-1}\Gamma_{g-p}(\beta)^{-1} \\ &\times \det(Y)^{\frac{g+1}{2}-\alpha-\beta}d_+(hY)^{\alpha-\frac{g+1}{2}+\frac{q}{4}}d_-(hY)^{\beta-\frac{g+1}{2}+\frac{p}{4}}\omega(2\pi Y; h, \alpha, \beta). \end{aligned} \tag{4.2}$$

Here $d_\pm(X)$ is the product of all positive (or negative) eigenvalues of X, $\Gamma_m(s) = \pi^{m(m-1)/4}\prod_{i=0}^{m-1}\Gamma(s - i/2)$ and $\omega(Y, h; \alpha, \beta)$ is an entire function on α and β.

Next we explain $S_g(\psi, h, s)$, which is called the (generalized) Siegel series. For any $T \in \mathrm{Sym}^g(\mathbb{Q})$ we can write

$$T = U\begin{pmatrix} \nu_1/\delta_1 & & \\ & \ddots & \\ & & \nu_g/\delta_g \end{pmatrix}V \quad U, V \in SL(g, \mathbb{Z}),\ (\nu_i, \delta_i) = 1,\ \delta_i > 0.$$

by the elementary divisor theorem. Put $\delta(T) = \prod \delta_i$, $\nu(T) = \prod \nu_i = \det(T)\delta(T)$. Now we define

$$S_g(\psi, h, s) = \sum_{\substack{T\in\mathrm{Sym}^g(\mathbb{Q}) \bmod 1 \\ p|\delta_i(T),\forall i}} \psi(\nu(T))\delta(T)^{-s}\mathbf{e}(hT), \tag{4.3}$$

which has the Euler product expression

$$S_g(\psi, h, s) = \prod_{q:\ \text{primes}} S_g^q(\psi, h, s)$$

with

$$S_g^q(\psi, h, s) = \begin{cases} \displaystyle\sum_{T \in \mathrm{Sym}^g(\mathbb{Q})_q \bmod 1} \psi(\delta(T))\delta(T)^{-s}\mathbf{e}(hT) & q \neq p; \\ \displaystyle\sum_{\substack{T \in \mathrm{Sym}^g(\mathbb{Q})_p \bmod 1 \\ p \mid \delta_i(T), \forall i}} \psi(\nu(T))\delta(T)^{-s}\mathbf{e}(hT) & q = p, \end{cases}$$

here $\mathrm{Sym}^g(\mathbb{Q})_q = \bigcup_n \frac{1}{q^n}\mathrm{Sym}^g(\mathbb{Z})$. It converges if $\mathrm{Re}(s) > g$, in particular $S_g(\psi, h, s)$ does not have a pole if $\mathrm{Re}(s) > g$ as explained above.

Remark. In the Fourier expansion of $E_{p,\psi}^k$, we substitute in the function ξ $\alpha = k + s$ and $\beta = s$, and study the behavior at $s = 0$. In the case of $k > g + 1$, the function ξ_g has zero if $h \not> 0$ thanks to the term $\Gamma_{g-p}(s)$. On the other hand the function S_g does not have a pole at $s = k > g + 1$, thus only $h > 0$ contributes to the Fourier coefficients, and in this case $\omega(2\pi Y, h; \alpha, 0) = 2^*\mathbf{e}(-2\pi hY)$.

If $q \neq p$, the local Siegel series $S_g^q(\psi, h, s)$ is already studied by many mathematicians for example Kaufhold, Siegel, Kitaoka, and finally Katsurada gives the explicit formula in [Kat]. We quote Kaufhold's result of degree 2.

Theorem 4.1 (Kaufhold).

$$\prod_{q \neq p} S_2^q(\psi, h, s) = \begin{cases} \dfrac{L(s-2,\psi)L(2s-3,\psi^2)}{L(s,\psi)L(2s-2,\psi^2)} & h = 0; \\[2ex] \dfrac{L(2s-3,\psi^2)}{L(s,\psi)L(2s-2,\psi^2)} \displaystyle\prod_{q \neq p} F_q & \mathrm{rank}\, h = 1; \\[2ex] \dfrac{L(s-1,\psi\chi_h)}{L(s,\psi)L(2s-2,\psi^2)} \displaystyle\prod_{q \neq p} G_q & \mathrm{rank}\, h = 2. \end{cases}$$

Here $L(s, \psi)$ denotes the Dirichlet L-function, χ_h is the quadratic character associated with $\mathbb{Q}(\sqrt{-\det 2h})/\mathbb{Q}$ and F_q and G_q are polynomials in q^{-s} depending on h, such that $F_q = G_q = 1$ for all but finite q.

Remark. In [Sh2] Shimura was interested in the holomorphy or the algebraicity of the Fourier coefficients. Then it suffices to consider twisted Eisenstein series $E_{p,\psi}^k|_k J_g(Z, s)$, whose Fourier coefficients are given by

$$p^{-2(k+2s)} \sum_{h \in \mathrm{Sym}^2(\mathbb{Z})^*} \left(\prod_{q \neq p} S_2^q(\psi, h, k+2s)\right) \xi_2(\frac{1}{p}Y, h, k+s, s)\, \mathbf{e}(\frac{hX}{p}).$$

In this case Kaufhold's results are enough to investigate the Fourier coefficients. Our aim is to give the explicit Fourier coefficients of $E^k_{p,\psi}(Z,s)$, thus we need to calculate $S^p(\psi,h,s)$.

5. Results

In this section we give an explicit formula for $S^p_2(\psi,h,s)$. There are three cases according to the rank of h. It suffices to consider the case for diagonal h; indeed there are natural bijection $\mathrm{Sym}^2(\mathbb{Q})_p \bmod 1 \simeq \mathrm{Sym}^2(\mathbb{Q}_p) \bmod \mathbb{Z}_p$, thus

$$S^p_2(\psi,h,s) = \sum_{\substack{T\in\mathrm{Sym}^2(\mathbb{Q}_p) \bmod \mathbb{Z}_p \\ p|\delta_i}} \psi(\nu(T))\delta(T)^{-s}\mathbf{e}(hT),$$

and for any $h \in \mathrm{Sym}^2(\mathbb{Q})$ there exists $M \in SL_2(\mathbb{Z}_p)$ such that $h[M]$ is diagonal.

Lemma 5.1.

$$S^p_2(\psi,0,s) = \begin{cases} 0 & \psi^2 \not\equiv 1; \\ \psi(-1)\,\dfrac{(p-1)p^{1-2s}}{1-p^{3-2s}} & \psi^2 \equiv 1, \psi \not\equiv 1; \\ \dfrac{p^{3-2s}(1+p^{1-s})}{(1-p^{2-s})(1-p^{3-2s})} & \psi \equiv 1. \end{cases}$$

Lemma 5.2. *Assume that ψ is a non-trivial character. Then for $h = \mathrm{diag}(t,0)$ with $\mathrm{ord}_p\, t = m$,*

$$S_2(\psi,h,s) = \begin{cases} 0 & \psi^2 \not\equiv 1; \\ a(p^{-s}) + \dfrac{b(p^{-s})}{1-p^{3-2s}} & \psi^2 \equiv 1. \end{cases}$$

Here $a(p^{-s})$ and $b(p^{-s})$ are polynomial in p^{-s} defined by

$$a(p^{-s}) = \psi(-1)\frac{p-1}{p^2}\sum_{k=1}^{m+1} p^{(3-2s)k},$$
$$b(p^{-s}) = \psi(-1)(p-1)p^{(3-2s)m+4-4s}.$$

Lemma 5.3. *Let $G(\psi)$ be the Gaussian sum of ψ, $\chi_p = (\frac{\cdot}{p})$. The value ε_p is defined by $G(\chi_p) = \varepsilon_p\sqrt{p}$. If $h = p^m\,\mathrm{diag}(\alpha, p^k\beta)$, $(p,\alpha\beta) = 1$ then $S^p_2(\psi,h,s) = S_1 + S_2$ with*

$$S_1 = \begin{cases} \sum_{k=1}^m p^{(3-2s)k-1} - \varepsilon_p^2 p & \text{if } \psi = \chi_p \text{ and } t = 0, \\ \sum_{k=1}^m p^{(3-2s)k-1} + (p-1)\varepsilon_p^2 p & \text{if } \psi = \chi_p \text{ and } t \geq 1, \\ \bar{\psi}\chi_p(\alpha\beta)G(\psi\chi_p)\varepsilon_p\sqrt{p} & \text{if } \psi \neq \chi_p \text{ and } t = 0, \\ 0 & \text{if } \psi \neq \chi_p \text{ and } t \geq 1. \end{cases}$$

$$S_2 = \begin{cases} 0 & \textit{if } t = 0 \\ \varepsilon_p^2 p^{-(2m+2)s+3m+1}\{(p-1)\sum_{n=1}^{\frac{t-2}{2}} p^{(3-2s)n} - p^{(3-2s)t/2}\} & \textit{if } \psi = \chi_p,\ t \geq 2 \textit{ is even,} \\ \varepsilon_p p^{-(2m+2)s+3m+1} & \\ \times\{p^{(3-2s)t+1/2}\bar{\psi}(\alpha\beta) + \varepsilon_p(p-1)\sum_{n=1}^{\frac{t-1}{2}} p^{(3-2s)n}\} & \textit{if } \psi = \chi_p,\ t \textit{ is odd,} \\ p^{-(2m+2+t)s+3m+(3t+3)/2}\varepsilon_p\bar{\psi}\chi_p(\alpha\beta)G(\psi)G(\psi\chi_p) & \textit{if } \psi \neq \chi_p,\ t \geq 2 \textit{ is even,} \\ p^{-(2m+2+t)s+3m+(3t+1)/2}\bar{\psi}(\alpha\beta)G(\psi)^2 & \textit{if } \psi \neq \chi_p,\ t \textit{ is odd.} \end{cases}$$

Gathering the above lemmas, we can give the explicit formula for the Fourier expansion of $E^k_{p,\psi}(Z,s)$.

Remark. Y. Mizuno [Miz] gave the Fourier expansion of $E^k_{p,\psi}(Z)$ for $k \geq 4$ in another way (Koecher-Maass lift of the Jacobi Eisenstein series).

Outline of the proof. Our first strategy is to rewrite the element of $T \in \mathrm{Sym}^2(\mathbb{Q})_p$ by symmetric co-prime pair. For $C, D \in M_g(\mathbb{Z})$, we say C and D are *symmetric* if $C\,{}^tD = D\,{}^tC$ and *co-prime* if there exist $X, Y \in M_g(\mathbb{Z})$ such that $CX + DY = 1_g$. Let

$$\mathcal{M}_g = \{(C,D) \in M_{g,2g}(\mathbb{Z}) \mid C, D \text{ are symmetric and co-prime}, \det C \neq 0\}.$$

Then we have the one to one correspondence between $GL_g(\mathbb{Z})\backslash\mathcal{M}_g$ and $\mathrm{Sym}^g(\mathbb{Q})$ by $(C,D) \mapsto C^{-1}D$, and

$$\delta(C^{-1}D) = |\det C|, \quad \nu(C^{-1}D) = \pm\det D.$$

We set

$$\mathcal{M}_g^p = \{(C,D) \in \mathcal{M}_g \mid \det C = p^a, C \equiv 0 \bmod p\},$$

and

$$\widetilde{\mathcal{M}}_g^p = \{(C,D) \in M_{g,2g}(\mathbb{Z}) \mid \det C = p^a,\ C \equiv 0 \bmod p,\ C\,{}^tD = D\,{}^tC\}.$$

In $\widetilde{\mathcal{M}}_g^p$ we only require the symmetric condition. The important fact is:

(*) For symmetric pair (C,D) with $\det C \neq 0$, we have $C = MC'$, $D = MD'$ with $(C',D') \in \mathcal{M}_g$.

Now we can write

$$\begin{aligned} S_2^p(\psi,h,s) &= \sum_{\substack{C \\ (C,D)\in SL(2,\mathbb{Z})\backslash\mathcal{M}_2^p}} \sum_{D \bmod C} \psi(\det D)(\det C)^{-s}\mathbf{e}(hC^{-1}D), \\ &= \sum_{\substack{C \\ (C,D)\in SL(2,\mathbb{Z})\backslash\widetilde{\mathcal{M}}_2^p}} \sum_{D \bmod C} \psi(\det D)(\det C)^{-s}\mathbf{e}(hC^{-1}D). \end{aligned}$$

The second equation follows from (*), for if (C,D) are not co-prime we can write $C = MC'$ and $D = MD'$; however $\det M$ must be divisible by p, $\psi(\det D) = \psi(\det MD') = 0$.

Now we study the set $\{(C, D \bmod C) \mid (C,D) \in SL(2,\mathbb{Z})\backslash\mathcal{M}_2^p\}$. Let $T(k,l) = \mathrm{diag}(p^k, p^{k+l})$. Then by the elementary divisor theorem, C runs thorough the set $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{Z})T(k,l)SL(2,\mathbb{Z})$ with $k \geq 1$, $l \geq 0$. If $l = 0$ a representative set is $T(k,0)$ only, while if $l \geq 1$, it is given by

$$\left\{T(k,l)V \,\middle|\, V = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix},\ u \in \mathbb{Z}/p^l\mathbb{Z}\right\} \cup \left\{T(k,l)V \,\middle|\, V = \begin{pmatrix} pu & 1 \\ -1 & 0 \end{pmatrix},\ u \in \mathbb{Z}/p^{l-1}\mathbb{Z}\right\}.$$

For such $C = T(k,l)V$, $D \bmod C$ runs through the set

$$\left\{\begin{pmatrix} a & b \\ p^l b & d \end{pmatrix} {}^tV^{-1} \,\middle|\, a,b \in \mathbb{Z}/p^k\mathbb{Z},\ d \in \mathbb{Z}/p^{k+l}\mathbb{Z}\right\}.$$

We shall prove Lemma 5.2 only. Othere cases follows from the similar calculation. Let $h = \mathrm{diag}(t,0)$, $t = p^m t'$ with $(t',p) = 1$ and $h' = \mathrm{diag}(t',0)$. Then

$$S_2^p(\psi,h,s) = \sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\sum_{V} \frac{1}{p^{(2k+l)s}} \sum_{\substack{a,b\in\mathbb{Z}/p^k\mathbb{Z} \\ d\in\mathbb{Z}/p^{k+l}\mathbb{Z}}} \psi(ad - p^l b^2)\mathbf{e}\left(\frac{1}{p^{k-m}}\begin{pmatrix} a & b \\ b & dp^{-l} \end{pmatrix}(h'[V^{-1}])\right).$$

Let us decompose the summation with respect to l and V.

$\boxed{l=0}$ In this case $V = 1_2$. The summation is

$$\sum_{k=1}^{\infty} p^{-2ks} \sum_{a,b,d\in\mathbb{Z}/p^k} \psi(ad-b^2)\mathbf{e}\left(\frac{t'a}{p^{k-m}}\right)$$

(change $a \mapsto pa_1 + a$, $b \mapsto pb_1 + b$, $d \mapsto pd_1 + d$)

$$= \sum_{k=1}^{\infty} p^{-2ks} \sum_{a_1,b_1,d_1\in\mathbb{Z}/p^{k-1}} \mathbf{e}\left(\frac{t'a_1}{p^{k-m-1}}\right) \sum_{a,b,d\in\mathbb{Z}/p} \psi(ad-b^2)\mathbf{e}\left(\frac{t'a}{p^{k-m}}\right)$$

(the first summation remains only $k \leq m+1$)

$$= \sum_{k=1}^{m+1} p^{-2ks+3k-3} \sum_{a,b,d\in\mathbb{Z}/p} \psi(ad-b^2)\mathbf{e}\left(\frac{t'a}{p^{k-m}}\right).$$

For the summation of a, b and d, if $a = 0$ then

$$\sum_{b,d} \psi(-b^2) = \begin{cases} 0 & \psi^2 \not\equiv 1, \\ \psi(-1)p(p-1) & \psi^2 \equiv 1, \end{cases}$$

while if $a \neq 0$ we can change the valuable $d \mapsto d + a^{-1}b^2$ and

$$\sum_{a\neq 0,b,d} \psi(ad)\mathbf{e}\left(\frac{t'a}{p^{k-m}}\right) = 0.$$

Hence $l=0$ part is

$$\begin{cases} 0 & \psi^2 \not\equiv 1; \\ \psi(-1)(p-1)p^{-2}\sum_{k=1}^{m+1} p^{(1-2s)k} & \psi^2 \equiv 1. \end{cases}$$

$\boxed{l \geq 1,\ V = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)}$. The summation is

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} p^{-(2k+l)s} \sum_{u \in \mathbb{Z}/p^l} \sum_{\substack{a,b \in \mathbb{Z}/p^k \\ d \in \mathbb{Z}/p^{k+l}}} \psi(ad)\mathbf{e}\left\{\frac{t'}{p^{k-m}}(a - 2ub + \frac{u^2 d}{p^l})\right\}. \tag{5.1}$$

Then the summation with respect to a:

$$\sum_{a \in \mathbb{Z}/p^k} \psi(a)\mathbf{e}\left(\frac{t'a}{p^{k-m}}\right) = \sum_{a_1 \in \mathbb{Z}/p^{k-1}} \mathbf{e}\left(\frac{t'a_1}{p^{k-m-1}}\right) \sum_{a \in \mathbb{Z}/p} \psi(a)\mathbf{e}\left(\frac{t'a}{p^{k-m}}\right)$$

remains only when $k = m+1$ and equals to $p^m G(\psi)$. Thus

$$(5.1) = G(\psi)\sum_{l=1}^{\infty} p^{-(2m+2+l)s+m} \sum_{u \in \mathbb{Z}/p^l} \sum_{\substack{b \in \mathbb{Z}/p^{m+1} \\ d \in \mathbb{Z}/p^{m+1+l}}} \psi(d)\mathbf{e}\left\{\frac{t'}{p}(-2ub + \frac{u^2 d}{p^l})\right\}$$

(looking at the summation for b, it remains only $p|u$ so we change $u \mapsto pu$)

$$= G(\psi)\sum_{l=1}^{\infty} p^{-(2m+2+l)s+2m+1} \sum_{u \in \mathbb{Z}/p^{l-1}} \sum_{d \in \mathbb{Z}/p^{m+1+l}} \psi(d)\mathbf{e}\left(\frac{u^2 d}{p^{l-1}}\right).$$

The famous formula for the Gaussian sum shows

$$\sum_{u \in \mathbb{Z}/p^{l-1}} \mathbf{e}\left(\frac{u^2 d}{p^{l-1}}\right) = \begin{cases} \chi_p(d)p^{(l-2)/2}G(\chi_p) & l \text{ is even}, \\ p^{(l-1)/2} & l \text{ is odd}. \end{cases}$$

Thus

$$\begin{aligned} (5.1) &= G(\psi)G(\chi_p)\sum_{l=1}^{\infty} p^{-2(m+1+l)s+2m+l} \sum_{d \in \mathbb{Z}/p^{m+2l+1}} \chi_p\psi(d) \\ &= \begin{cases} 0 & \psi \neq \chi_p \\ \psi(-1)(p-1)\sum_{l=1}^{\infty} p^{-2(m+l+1)s+3m+3l+1} & \psi = \chi_p. \end{cases} \end{aligned}$$

The lower term is nothing but $b(p^{-s})(1-p^{3-2s})^{-1}$

$\boxed{l \geq 1,\ V = \left(\begin{smallmatrix} pu & 1 \\ -1 & 0 \end{smallmatrix}\right)}$. One can show similarly that this part vanishes.

We conclude the proof of Lemma 5.2. □

6. Dimensions of the space of Eisenstein series

As an application of the previous section, we calculate the dimensions of the space of Eisenstein series in low weight case, i.e. $k = 1, 2, 3$. First it is already known in the case $k = 1$.

Theorem 6.1 (G.).

$$\dim \mathcal{E}_1(\Gamma^2(p)) = \begin{cases} \frac{1}{2}(p^2+1) & p \equiv 3 \bmod 4, \\ 0 & p \equiv 1 \bmod 4. \end{cases}$$

Hence it suffices to consider the case $k = 2$ or 3.

Remark. In the proof of Theorem 6.1, the author use the theta series to construct the element of $\mathcal{E}_1(\Gamma^2(p))$. In particular $\mathcal{E}_1(\Gamma_0^2(p), \psi) = 0$ if $\psi^2 \not\equiv 1$.

6.1. **The case of weight 3.** Let $k = 3$. By (4.2), Theorem 4.1, 5.1, 5.2 and 5.3 we can prove the following result in another way, i.e. using the Fourier expansion (4.1).

Theorem 6.2 (Shimura). *For any $\psi(-1) = -1$, $E^3_{p,\bar{\psi}}(Z) := E^3_{p,\bar{\psi}}(Z, 0) \in M_k(\Gamma_0^2(p), \psi)$. Moreover $C_0(E^3_{p,\psi}) = 1$, $C_0(E^3_{p,\psi}|_3 J_2) = 0$.*

As far as the author knows, there were no assertion for $C_0(E^3_{p,\psi})$ before. Now the main result of this subsection is as follows.

Theorem 6.3. *Let p be an odd prime.*

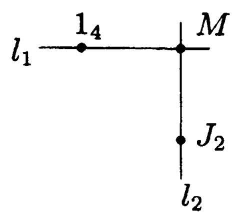
$$\dim \mathcal{E}_3(\Gamma^2(p)) = \frac{1}{2}(p^4 - 1).$$

First we shall show the following.

Theorem 6.4.

$$\dim \mathcal{E}_3(\Gamma_0^2(p), \psi) = \begin{cases} 3 & \psi^2 \equiv 1, \\ 2 & \psi^2 \not\equiv 1. \end{cases}$$

Outline of the proof of Theorem 6.4. The structure of the boundary of the Satake compactification of $\Gamma_0^2(p)\backslash\mathfrak{H}_2$ is as follows:



Here

$$M = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We explain the meaning of the figure. The lines l_1 and l_2 represent the modular curves $\Gamma_0^1(p)\backslash\mathfrak{H}_1$ and $\Gamma_0^1(p)^{J_1}\mathfrak{H}_1$ (with $\Gamma_0^1(p)^{J_1} = J_1^{-1}\Gamma_0^1(p)J_1$) respectively. Both of modular curves have 2 cusps ∞ and 0. These modular curves intersect at both of the cusp 0, which also corresponds to the 0-dimensional cusp M of $\Gamma_0^2(p)\backslash\mathfrak{H}_2$.

The above figure shows $\dim \mathcal{E}_3(\Gamma_0^2(p), \psi) \leq 3$.

Lemma 6.5 ([Gu, Lemma 3.7]). *Let $\psi^2 \not\equiv 1$. For any $f \in M_k(\Gamma_0^2(p), \psi)$, we have $C_0(f|_k M) = 0$.*

Thus if $\psi^2 \not\equiv 1$, we have $\dim \mathcal{E}_3(\Gamma_0^2(p), \psi) \leq 2$. Put

$$F_{p,\psi}^3(Z) := \sum_{T \in \mathrm{Sym}^2(\mathbb{F}_p)} E_{p,\psi}^3|_3\gamma(T), \quad \gamma(T) = \begin{pmatrix} 0 & 1_2 \\ -1_2 & T \end{pmatrix}.$$

Then $F_{p,\psi}^3(Z) \in M_3(\Gamma_0^2(3), \psi)$. We can calculate the value of $E_{p,\psi}^3$ and $F_{p,\psi}^3$ at each cusp:

$$C_0(E_{p,\bar{\psi}}^3|_3\gamma) = \begin{cases} 1 & \gamma = 1_4 \\ 0 & \gamma = M, J_2, \end{cases} \qquad C_0(F_{p,\psi}^3|_3\gamma) = \begin{cases} 1 & \gamma = J_2 \\ 0 & \gamma = 1_4, M. \end{cases}$$

Thus if $\psi^2 \not\equiv 1$,

$$\dim \mathcal{E}_3(\Gamma_0^2(p), \psi) = 2.$$

Next we consider the case $\psi^2 \equiv 1$. We need to know the valu $C_0(E_{p,\psi}^3|_3M)$, however if one consider the Fourier expansion of $E_{p,\psi}^3|_3M$, then the "Siegel series" does not have the Euler product expression. We use the following technique. Let Φ be the Siegel-operator: for $z \in \mathfrak{H}_1$, $f \in M_k(\Gamma_0^2(p), \psi)$,

$$\Phi(f)(z) = \lim_{\lambda \to \infty} f\left(\begin{pmatrix} z & 0 \\ 0 & i\lambda \end{pmatrix}\right) \in M_k(\Gamma_0^1(p), \psi).$$

Siegel-operator is nothing but the restriction of the Siegel modular forms to the 1-dimensional cusp of the Satake compactification. The above figure shows

$$C_0(f|_k M) = C_0(\Phi(f)|_k J_1), \ \forall f \in M_k(\Gamma_0^2(p), \psi).$$

We can calculate the Fourier expansion of $\Phi(E_{p,\psi}^3(Z))$ using the result of previous section, especially Lemma 5.2, and write $\Phi(E_{p,\psi}^3(Z))$ by using elliptic Eisenstein series. Thus we know the Fourier expansion of $\Phi(E_{p,\psi}^3)|_1 J_1$, and finally get $C_0(E_{p,\psi}^3(Z)) = 0$.

Now put

$$G := \sum_{c_1, d_2 \in \mathbb{Z}/p} E_{p,\psi}^3|_3\alpha(c_1, d_2) + \sum_{d_1 \in \mathbb{Z}/p} E_{p,\psi}^3|_3\beta(d_1), \in M_3(\Gamma_0^2(p), \psi)$$

with

$$\alpha(c_1,d_2)=\begin{pmatrix}0&0&0&-1\\-1&0&0&0\\c_1&1&0&d_2\\0&0&-1&c_1\end{pmatrix},\ \beta(d_1)=\begin{pmatrix}0&0&-1&0\\0&1&0&0\\1&0&d_1&0\\0&0&0&1\end{pmatrix},$$

then

$$C_0(G^3|_3\gamma)=\begin{cases}1 & \gamma=M,\\ 0 & \gamma=1_4, J_2.\end{cases}$$

Thus $E^3_{p,\psi}$, $F^3_{p,\psi}$ and $G^3_{p,\psi}$ are linearly independent, which shows $\dim\mathcal{E}_3(\Gamma^2_0(p),\psi)=3$. □

Theorem 6.3 follows from Theorem 6.4 and the theory of the representations of finite groups. We can show that $\dim\mathcal{E}_3(\Gamma^2_0(p),\psi)$ equals to the number of the irreducible representation of $Sp(2,\mathbb{F}_p)$, which appears $\mathcal{E}_3(\Gamma^2(p))$. For the details see [Gu].

6.2. **The case of weight 2.**

Theorem 6.6 (Shimura). *Assume* $\psi(-1)=1$. *If* $\psi^2\not\equiv 1$, $E^2_{p,\psi}(Z)=E^2_{p,\psi}(Z,0)\in M_2(\Gamma^2_0(p),\psi)$. *Moreover* $C_0(E^2_{p,\psi}(Z))=1$, $C_0(E^2_{p,\psi}|_2J_2(Z))=0$.

As is similar to the case of degree 3, we can show that if $\psi^2\not\equiv 1$,

$$\dim\mathcal{E}_2(\Gamma^2_0(p),\psi)=2.$$

Let $\psi^2\equiv 1$. Unfortunately in this case $E^2_{p,\psi}(Z,0)$ is not holomorphic in Z. However using the result by Boecherere and Schmidt [BS], we can construct the Eisenstein series. Put

$$\widetilde{E}^2_{p,\psi}(Z,s)=C\,L(2+2s,\psi)L(2+4s,\psi^2)\det(Y)^sE^2_{p,\psi}(Z,s),$$

with some normalizing constant C. Then by [BS, Proposition 5.2. b)]

$$\widetilde{E}^2_{p,\psi}(Z):=\widetilde{E}^2_{p,\psi}(Z,-1/2)\in M_2(\Gamma^2_0(p),\psi).$$

Let $\psi\equiv 1$. We use the following fact of the ellptic modular forms: $\dim\mathcal{E}_2(\Gamma^1_0(p))=1$ and a basis f take non-zero value at both cusps 0 and ∞. Then the figure of the boundary shows

$$\dim\mathcal{E}_2(\Gamma^2_0(p))=1.$$

Finally consider the case $\psi=(\frac{\cdot}{p})=\chi_p$, which occures only when $p\equiv 1 \bmod 4$, since ψ is assumed to be even. We have three elements in $\mathcal{E}_2(\Gamma^2_0(p),\chi_p)$: $\widetilde{E}^2_{p,\psi}$, $\widetilde{F}^2_{p,\psi}$, $\widetilde{G}^2_{p,\psi}$ like weight 3 case. However

$$C_0(\widetilde{E}^2_{p,\psi}|_2\gamma)=\begin{cases}1 & \gamma=1_4,\\ 0 & \gamma=M,\\ -\dfrac{1}{p^2} & \gamma=J_4,\end{cases}\quad C_0(\widetilde{F}^2_{p,\psi}|_2\gamma)=\begin{cases}-p & \gamma=1_4,\\ 0 & \gamma=M,\\ \dfrac{1}{p} & \gamma=J_4,\end{cases}$$

and

$$C_0(\widetilde{G}^2_{p,\psi}|_2\gamma) = 0 \text{ for all } \gamma.$$

Thus we can get only 1 element in $\mathcal{E}_2(\Gamma_0^2(p), \chi_p)$.

To get other elements, we use the theory of theta series. There exist $Q \in M_4(\mathbb{Z})$ of even positive definite with $\det Q = p$. Put$Q' = pQ^{-1}$. Then the theta series is defined by

$$\theta^Q(Z) = \sum_{N \in M_{2,4}(\mathbb{Z})} \mathbf{e}(\frac{1}{2}Q[N]Z).$$

We have $\theta^Q(Z), \theta^{Q'}(Z) \in M_2(\Gamma_0^2(p), \chi_p)$ and

$$C_0(\theta^Q|_2\gamma) = \begin{cases} 1 & \gamma = 1_4, \\ -\dfrac{1}{\sqrt{p}} & \gamma = M, \\ \dfrac{1}{p} & \gamma = J_2, \end{cases} \quad C_0(\theta^{Q'}|_2\gamma) = \begin{cases} 1 & \gamma = 1_4, \\ -\dfrac{1}{p\sqrt{p}} & \gamma = M, \\ \dfrac{1}{p^3} & \gamma = J_2. \end{cases}$$

Now we get 3 elements $E^2_{p,\psi}$, θ^Q and $\theta^{Q'}$. However since

$$\det \begin{pmatrix} 1 & 1 & 1 \\ 0 & -\frac{1}{\sqrt{p}} & -\frac{1}{p\sqrt{p}} \\ -\frac{1}{p^2} & \frac{1}{p} & \frac{1}{p^3} \end{pmatrix} = 0.$$

these are linearly dependent in $\mathcal{E}_2(\Gamma_0^2(p), \chi_p)$, so we can only know

$$\dim \mathcal{E}_2(\Gamma_0^2(p), \psi) = 2 \text{ or } 3.$$

At present the authore can not determine which situation will occur. As a consequence we have

Theorem 6.7.

$$\dim \mathcal{E}_2(\Gamma_0^2(p), \psi) = \begin{cases} 2 & \psi^2 \not\equiv 1, \\ 1 & \psi \equiv 1, \\ 2 \text{ or } 3 & \psi = (\frac{\cdot}{p}). \end{cases}$$

Theorem 6.8. (1) *If* $p \equiv 3 \bmod 4$, *then*

$$\dim \mathcal{E}_2(\Gamma^2(p)) = \frac{1}{2}(p^2+1)(p^2-p-3).$$

(2) *If* $p \equiv 1 \bmod 4$, *then*

$$\dim \mathcal{E}_2(\Gamma^2(p)) = \frac{1}{2}(p^2+1)(p^2-p-3) \text{ or } \frac{1}{2}(p^2+1)(p^2-p-4).$$

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